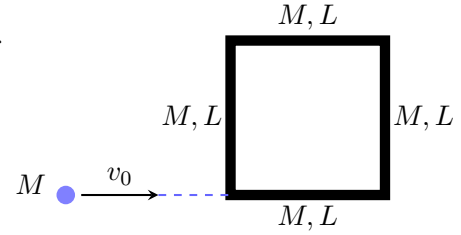


1. A rigid square is made by joining 4 homogenous rods of length L and mass M as shown in the figure. For a single rod the moment of inertia about its center of mass is $I_{CM} = ML^2/12$.



- [5 Pts.] (a) What is the moment of inertia of the square around an axis perpendicular to the plane of the page, and passing through its center of mass? (Call this quantity I_0 and continue the problem even if you cannot solve this part.)

For each rod, the distance from its center of mass to the center of the square is $L/2$. Hence, by the parallel-axis theorem,

$$I_{\text{rod}} = \frac{ML^2}{12} + M \left(\frac{L}{2} \right)^2 = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}.$$

There are four rods, so

$$I_0 = 4 \left(\frac{ML^2}{3} \right) = \frac{4}{3} ML^2.$$

The square is resting on a horizontal frictionless plane, a point of object of mass M (same as one of the rods forming the square) hits its corner with a velocity v_0 parallel to the bottom side of the square as shown in the figure. The collision is completely inelastic, the mass sticks to the corner of the square and objects move together after the collision

- [8 Pts.] (b) Which quantities are conserved in the collision (Correct choice +2, Wrong choice -1 for each)

	Conserved	Not Conserved
Mechanical Energy		✓
Linear Momentum	✓	
Angular Momentum with respect to the collision point	✓	
Angular Momentum with respect to the center of mass of the system	✓	

- [4 Pts.] (c) What is the speed of the center of mass of the system after the collision?

The final total mass is $M_{\text{tot}} = 5M$. By conservation of linear momentum,

$$Mv_0 = 5Mv_{\text{CM}} \Rightarrow v_{\text{CM}} = \frac{v_0}{5}.$$

- [8 Pts.] (d) What is the angular velocity of the system around its center of mass after the collision? Take the center of the square before collision as the origin. The point mass strikes the lower-left corner, whose position is

$$\vec{r}_p = \left(-\frac{L}{2}, -\frac{L}{2} \right).$$

The center of mass of the final system is

$$\vec{R}_{\text{CM}} = \frac{4M(0,0) + M\left(-\frac{L}{2}, -\frac{L}{2}\right)}{5M} = \left(-\frac{L}{10}, -\frac{L}{10}\right).$$

Thus the position of the impact point relative to the final center of mass is

$$\vec{r} = \vec{r}_p - \vec{R}_{\text{CM}} = \left(-\frac{2L}{5}, -\frac{2L}{5}\right).$$

The moment of inertia of the final system about the final center of mass.

The square contributes

$$I_{\text{square}} = I_0 + 4Md^2 = \frac{4}{3}ML^2 + \frac{2}{25}ML^2,$$

where

$$d = \sqrt{\left(\frac{L}{10}\right)^2 + \left(\frac{L}{10}\right)^2} = \frac{\sqrt{2}L}{10}.$$

The stuck point mass contributes

$$I_{\text{point}} = Mr^2 = \frac{8}{25}ML^2,$$

where

$$r = \sqrt{\left(\frac{2L}{5}\right)^2 + \left(\frac{2L}{5}\right)^2} = \frac{2\sqrt{2}L}{5}.$$

Hence the total moment of inertia is

$$I_{\text{tot}} = \frac{4}{3}ML^2 + \frac{2}{25}ML^2 + \frac{8}{25}ML^2 = \frac{26}{15}ML^2.$$

Conservation of angular momentum about the final center of mass gives

$$\frac{2}{5}MLv_0 = \frac{26}{15}ML^2\omega.$$

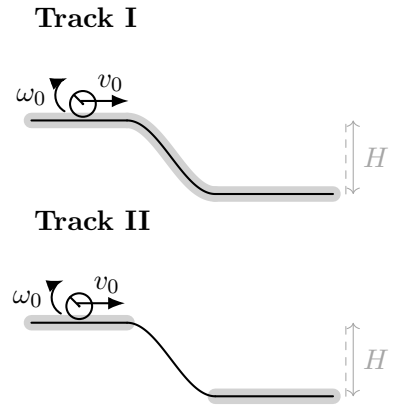
Solving for ω ,

$$\omega = \frac{3v_0}{13L}.$$

2. A solid cylinder of mass m , radius R , and moment of inertia $I = \frac{1}{2}mR^2$ is initially moving on the upper horizontal part of a track with center-of-mass speed v_0 and angular speed $\omega_0 = \frac{v_0}{R}$, so that it is rolling without slipping. The upper and lower horizontal parts of the track are separated by a vertical distance H , as shown in the figure. Take the gravitational acceleration to be g .

Two different tracks are considered:

- **Track I:** Friction is present everywhere on the track in such a way that the cylinder rolls without slipping throughout its motion.
- **Track II:** The upper and lower horizontal parts are rough enough for rolling without slipping, but the curved connecting part is frictionless.



[8 Pts.]

- (a) Find the final speed of the cylinder after it reaches the lower horizontal part of **Track I**. For Track I, the cylinder rolls without slipping throughout the motion. Therefore, $v = \omega R$ at all times.

Conservation of energy:

$$\frac{1}{2}mv_0^2 + \frac{1}{2}I\omega_0^2 + mgH = \frac{1}{2}mv_1^2 + \frac{1}{2}I\omega_1^2.$$

Since rolling without slipping continues, $\omega_1 = \frac{v_1}{R}$.

Hence,

$$\frac{3}{4}mv_0^2 + mgH = \frac{3}{4}mv_1^2 \Rightarrow v_1 = \sqrt{v_0^2 + \frac{4}{3}gH}$$

[10 Pts.]

- (b) Find the final speed of the cylinder after it has moved on the lower horizontal part of **Track II** long enough to again roll without slipping.

For Track II, the curved part is frictionless. Therefore no torque acts on the cylinder while it moves along the curved section, so the angular speed remains constant: $\omega = \omega_0 = \frac{v_0}{R}$

The translational motion gains gravitational potential energy:

$$\frac{1}{2}mv_0^2 + mgH = \frac{1}{2}mv_2^2.$$

Hence,

$$v_2^2 = v_0^2 + 2gH \Rightarrow v_2 = \sqrt{v_0^2 + 2gH},$$

When the cylinder reaches the lower rough horizontal surface, slipping occurs until rolling without slipping is re-established.

$$mv_2R + I\omega_0 = mv_fR + I\omega_f.$$

$$mv_2R + \frac{1}{2}mR^2\frac{v_0}{R} = mv_fR + \frac{1}{2}mR^2\frac{v_f}{R}.$$

Simplifying,

$$v_f = \frac{2}{3}v_2 + \frac{1}{3}v_0 \Rightarrow v_f = \frac{2}{3}\sqrt{v_0^2 + 2gH} + \frac{1}{3}v_0.$$

[7 Pts.] (c) Which final speed is larger? Explain physically.

Compare v_1 and v_f .

We have

$$v_1^2 = v_0^2 + \frac{4}{3}gH,$$

while

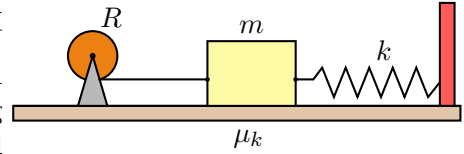
$$v_f^2 = \frac{5}{9}v_0^2 + \frac{8}{9}gH + \frac{4}{9}v_0\sqrt{v_0^2 + 2gH}$$

One finds

$$v_1 > v_f.$$

Physically, in Track II some mechanical energy is lost during the slipping process before pure rolling is restored. In Track I there is no slipping, so no energy is dissipated.

3. A block of mass m is on a horizontal surface and is connected on its right to a spring of spring constant k . On its left, the block is attached to a light string wrapped around a pulley of radius R . The pulley has moment of inertia I about its frictionless axis. The coefficient of kinetic friction between the block and the surface is μ_k . Initially, the spring is stretched by a distance x_0 from its natural length, and the system is released from rest. Assume the string does not slip on the pulley. (Gravitational acceleration is g .)



[10 Pts.]

- (a) Find the speed of the block when the spring first reaches its natural length. (Call this quantity v_0 and continue the problem even if you cannot solve this part.) Initially the spring is stretched by a distance x_0 , so the initial spring potential energy is

$$U_i = \frac{1}{2}kx_0^2.$$

As the block moves toward the unstretched position, kinetic friction does negative work:

$$W_f = -\mu_k mgx_0.$$

The final kinetic energy is

$$K_f = \frac{1}{2}mv_0^2 + \frac{1}{2}I\omega^2.$$

Using energy conservation with friction and $\omega = v_0/R$,

$$\frac{1}{2}kx_0^2 - \mu_k mgx_0 = \frac{1}{2} \left(m + \frac{I}{R^2} \right) v_0^2.$$

Hence,

$$v_0 = \sqrt{\frac{kx_0^2 - 2\mu_k mgx_0}{m + \frac{I}{R^2}}}.$$

[5 Pts.]

- (b) Find the angular speed ω of the pulley at that instant. Since the string does not slip,

$$\omega = \frac{v_0}{R} = \sqrt{\frac{kx_0^2 - 2\mu_k mgx_0}{mR^2 + I}}.$$

[5 Pts.]

- (c) Determine the minimum value of x_0 required for the block to actually reach the unstretched position.

The minimum value of x_0 occurs when the block barely reaches the unstretched position, so $v_0 = 0$. Then the final kinetic energy is zero, and energy conservation gives

$$\frac{1}{2}kx_0^2 - \mu_k mgx_0 = 0 \quad \Rightarrow \quad x_{0,\min} = \frac{2\mu_k mg}{k}.$$

Note: In a real system, it can reach this position a little earlier due to friction, but this is neglected in the above solution. The students noticed that get an extra point.

[5 Pts.]

- (d) the maximum compression x_1 of the spring after the block passes the unstretched position. (Note that the string can pull, but cannot push!)

Pulley rotates with same ω . Energy conservation gives

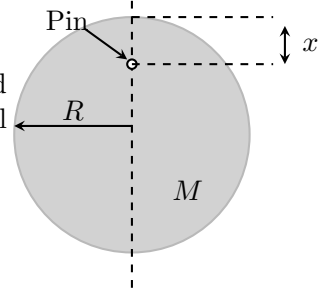
$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx_1^2 + \mu_k mgx_1.$$

$$x_1^2 + \frac{2\mu_k mg}{k}x_1 - \frac{m}{k}v_0^2 = 0.$$

Solving the quadratic equation,

$$x_1 = -\frac{\mu_k mg}{k} + \sqrt{\frac{\mu_k^2 m^2 g^2}{k^2} + \frac{mv_0^2}{k}}.$$

4. A homogeneous disk of radius R and mass M has a small hole drilled through it at a distance x from its edge. The disk is hung from the wall by means of a pin through the hole, and used as a pendulum. ($I_{CM} = \frac{1}{2}MR^2$).



[15 Pts.]

- (a) What is the period of this pendulum for small oscillations? The disk is a physical pendulum. The pivot is located a distance x from the edge of the disk, so the distance from the pivot to the center of mass is $d = R - x$. The moment of inertia about the pivot is found using the parallel-axis theorem:

$$I_p = I_{CM} + Md^2 = \frac{1}{2}MR^2 + M(R - x)^2.$$

$$\tau = I_p \alpha \Rightarrow I_p \frac{d^2\theta}{dt^2} = -Mgd \sin \theta.$$

For small oscillations, $\sin \theta \simeq \theta$. Therefore,

$$\frac{d^2\theta}{dt^2} + \frac{Mgd}{I_p} \theta = 0.$$

Thus,

$$\omega^2 = \frac{Mgd}{I_p} = \frac{g(R - x)}{\frac{1}{2}R^2 + (R - x)^2}.$$

Hence the period is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R^2 + 2(R - x)^2}{2g(R - x)}}.$$

[10 Pts.]

- (b) For what value of x is the period shortest? To find the value of x for which the period is shortest, minimize T ,

$$\frac{dT}{dx} = 0 \Rightarrow 2\pi \left(\frac{R^2 + 2(R - x)^2}{2g(R - x)} \right)^{-1/2} \left(\frac{R^2 - 2(R - x)^2}{2g(R - x)^2} \right) = 0.$$

Hence,

$$x = R \left(1 - \frac{1}{\sqrt{2}} \right).$$

The minimum value of the period is

$$T_{\min} = 2\pi \sqrt{\frac{\sqrt{2}R}{g}}.$$